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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

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FROM FAR AND NEAR

Recently it occured to us to raise this question: To what extent, have the lines of development in modern science, particularly in mathematics, been determined by mere chance? Is it conceivable, for example, that sets of foundation principles in geometry, algebra, analysis, fundamentally different from those in existence might have resulted had the *initial* trends in mathematics been denied the impress of such minds as Euclid, Apollonius, Descartes, Fermat, Galois, Newton, Leibnitz, Gauss, Euler and many others.

The question is too serious for easy settlement but we believe that the average instinct is prone to answer it in the negative. There are too many indications in history that every science whose growth is measured by the centuries must be the product, not of the building effort of a single mind, possibly inspired by the accident of some circumstance, but of a multi-minded race intelligence. Each generation must build on the amassed results of the previous generation.

The raising of the above question and our comment upon it were inspired by some recent notations in our reading. We have observed that, curiously enough, of late a number of

writers, writing without collaboration and, evidently, from widely different view-ponts, have made the *same* criticism of the products of many of our schools and colleges, namely, the inability of the average graduate to *do independent* and *effective thinking*. Is such criticism the result of a multi-mindedness which is destined to enforce radical reform in school methods? When one listens below the level of the printed comment, say, to passing remark, many voices are heard to the same effect. They come from far and near.

Let us be frank and honest above all things, trying to see with clear eyes.

If the indictment is correct, then we mathematicians and teachers of mathematics stand indicted, too. For, if ever instrument was built or conceived for perfect adaptation to training in the art of thinking, that instrument is *mathematics*. No mathematician, no teacher of mathematics will deny it. Yet, if the indictment is just, we of the mathematics profession must stand more deeply indicted than the workers in any other field.

The gauntlet is thrown down!

Let us take stock of our professions!

Let us dig up from its mouldy discard where too many of us long ago cast it in hasty deference to half-baked psychological experiment, the doctrine of *mathematical discipline*, lining up our teaching programs with an unyielding assumption that the mind incapable of being taught to think by means of mathematics cannot be taught to think by any means!

—*S. T. S.*

THE FRESHMAN PROBLEM

By R. L. O'QUINN
Louisiana State University

After ten years experience in teaching mathematics at Louisiana State University, where I have always had a large percentage of freshmen, I have come to the conclusion that the problem of successfully teaching mathematics to freshmen presents a great number of difficulties, and must be approached from several different directions if we are to achieve any marked suc-

cess. From time to time the mathematics staff at Louisiana State University has made a more or less detailed study of the problem, and at one time collected a great amount of data concerning the percentages of failures in freshman mathematics, at different institutions throughout the country. We found in the course of our inquiry that these ranged from 30% to 56% at the various institutions.

Let me state emphatically to begin with, that it is not the purpose of this article to try to prove that we are failing too great a percentage of our freshmen, or that we should lower our standards in order that more might pass. However, let us rather see why they fail.

Mathematical training is such that it must come step by step in logical sequence, and weakness anywhere along the line may cause the student to be unable to do the work in the next grade or class. For this reason it is absolutely essential that in the high school training of the freshman he should have been required all along the route to think logically and clearly. College teachers will answer that this is a desideratum in other fields as well as in mathematics, but there is certainly no field where wrong habits of study and poor teaching can show their baneful effects as much as in mathematics. There are good teachers of high school mathematics, and there are poor teachers, just as in any other courses the student takes. However, in justice to the college teacher who fails 40% of his freshman mathematics students it must be said that at least 15% of these students were failed before they left their homes, either through lack of ability or through improper training.

We may not remedy the former, but we can the latter. It seems that superintendents and principals would uniformly insist on thorough training in mathematics before permitting one to teach it. In a great many states, teachers of mathematics in the high schools must have had at least two years of college work in mathematics, proving that they have both an interest and some ability in that field.

We must not assume that, while it may take a specialist in some other field, just anyone who has the work can teach high school mathematics. The ability to do mathematics is based on clear, concise, and logical deductions, and this ability is rarer

than a good many others. At least we cannot expect pupils to be trained to think correctly unless the teacher has such training, together with natural ability above the average. The most important thing learned in mathematics is how to think, and this must be kept uppermost in the mind of the teacher at all times.

But far more important to the teacher of college mathematics is how to handle properly the freshmen who enter the institution and to schedule his courses. Here there is a wide diversity of opinion, both as to the method of conducting the recitation, and as to what should be expected of the beginner in freshman mathematics. The average freshman comes from a small school, perhaps located in the country or in a small town, where he is known intimately by his teachers, and where he receives constantly their advice and encouragement. His work has been always under the eye of the teacher, and he is surrounded by his family and friends. When he enters the university he is plunged into a new environment, is thrown at once on his own resources, and in most cases he is taught to a great extent by the lecture method. Is it any wonder that many become discouraged, lose interest, and fail the first semester? Psychology plays an important part during the first part of his course, and keeping up the morale of the class is one of the most important elements in the successful handling of the class. If one gets acquainted with the members of his class as early as possible and shows a personal interest in them he has a good opportunity to give encouragement that may bear fruit. Personally, I have found that if I have a good many poor papers in the first test, it is a good plan to tell the class that the grades will not be recorded, but another test will be given later. The first grades, which I assign, even the failures, are never extremely low. Then the students must be made to understand that we expect and intend to have improvement, if they are to pass. But let us not be too quick to condemn, and assume that the student who does practically nothing on the first test is an out-and-out dumbbell. Perhaps if we could get away from the lecture method, and have closer supervision of work done we might accomplish more.

In his celebrated "Discourse on Method", Descartes says that there is no problem so intricate that it cannot be made simpler by resolving the difficulty into its different components,

and attacking each in turn, and that the process can be carried on by successive steps until we make the solution depend upon something that is comparatively simple. This, indeed seems to have been the method of the great teachers of all ages. We learn to think by thinking.

It is far easier for a teacher to use the lecture method exclusively and to tell the students the subject matter of the lesson. But to teach the student to think we should anticipate his difficulties, and then by skillful questioning lead him to discover for himself a method of overcoming the difficulty. Since habits of thinking soon become formed, it is most imperative that the freshman should from the beginning be taught to use his native powers to the best advantage in order that he may be able to constantly increase his ability to do logical and purposeful thinking, with increasing profit and pleasure to himself.

A MATHEMATICS CLUB

By PROFESSOR C. D. SMITH
A. and M. College, Mississippi.

We are organizing a Mathematics Club which in some respects differs from many clubs known by this name. The history of such clubs is all too often a brief statement of membership for everybody, too frequent meetings with little point of interest, a lag of enthusiasm, and finally a state of neglect or abandonment. Some of our number have faith to believe and a sufficient love of mathematics to hope that the type of organization which we have undertaken shall have a different history. We expect it to make a permanent and valuable contribution to the education of its members and that they shall have a large part in the building of a greater mathematics program for our college and state.

The charter membership of this club was selected from students who rank in the highest five per cent of those studying advanced mathematics during the year 1930-1931. This honor is awarded them because they have excelled. Other students of advanced mathematics who maintain a high standard of scholarship may be recommended by members of the club. No one can

be elected to membership who has as many as two votes cast against him. This method of selection is not only the highest honor that teachers and fellow-students can confer but it guarantees the type of member who will work for his club with a zealous enthusiasm characteristic of the true spirit of science. It is a Scholarship Society in which every member is anxious to work. Such a club must have unbounded possibilities.

Major objectives of the club may be stated briefly as follows:

1. To support the development of a greater and more useful mathematics program at the college.

2. To seek talented students who should develop special abilities for leadership in mathematics.

3. To study the relations of mathematics to science and industry.

4. The solution of problems.

5. Preparation of papers on mathematical subjects.

6. To assist in the preparation of card indices for mathematical reference in the college library.

7. To offer service to the public whenever information of a mathematical nature may be desired.

8. To foster special activities of interest to students of mathematics and cultivate a social tone which is a recognized asset to those who hold positions in society.

One need not comment on the values of such objectives. For one in the upper classes to seek and encourage freshmen who show ability for success in advanced subjects and to practice with his fellows the ability to use his education for the benefit of others surely constitutes a program of great prospect. Time and the ability of the members will be the only natural boundary for achievement.

A WORTH WHILE EXPERIMENT

By W. R. CLIVE,
L. P. I. Demonstration School, Ruston, La.

To the readers of the Mathematics News Letter, a story such as the following is certain to be of interest.

I have a small class (5 girls and 2 boys) in intermediate algebra. This class has been kicked around from pillar to post

for years. All teachers have found these pupils to be "dumb-bells." It has been the thing naturally expected that each of these children would be failures in all or most of their subjects from year to year. It is not my purpose to delve into the various causes for the above conditions. Suffice it to say, that these pupils all come from decent homes; each child is fairly neat, clean and, ordinarily, a good school citizen except for classroom work. Standardized tests show that the class contains no member quite up to norms and standards of general achievement and intelligence.

I was not happy to have the job of teaching this group, but decided to experiment. Personally, I had never taught the group before, as principal, I had known that they were not good students.

From the first day, I have taken the attitude of believing that "This class can do, and is doing mighty good work."

I assumed this attitude of believing in the ability of the class with the idea of encouraging them to really make an effort. In order to secure to them the self confidence, which they lacked, I have been very careful all along to confine the work to the degree of difficulty that the class is capable of handling; moving very slowly and carefully, at first; painstakingly explaining all the obscure points; jealously guarding the pupils from the feeling of defeat.

The pupils have seemingly been quick to catch the spirit of interest in the work. They assert a real liking for algebra; they are coming to believe themselves capable of solving hard problems. Often they must be cautioned and restrained, to prevent their "biting off more than they can chew."

My assignments always contain a statement of the amount of work to be required, with directions as to extra work that may be done by any ambitious enough to undertake it.

Almost without exception, I have found that on the following day, each pupil has done more than the minimum assignment required.

Most of the class time is spent by the pupils in work at the board. The pupils are told to choose any problems from the current exercises that they have had trouble with. The teacher's time is spent in checking the work being done and in giving any help that seems to be necessary.

The real teacher activity is confined to the assignment period in which new work is introduced and discussed. The word "*drill*" is avoided; the class periods are spent in *practice* on the work previously taught and studied.

The result of the above procedure is that the class dealt with is *not* an extra good class, but it is a far better class than its previous records would have predicted. The pupils are *happy* in the work, and feel that they are doing something worthwhile. Their work in other classes has shown a general improvement. There is no doubt that their having found success in one class, has improved their mental attitude toward school work in general. Thus, a part of their general improvement can be attributed to their work in algebra.

I think that we can draw some conclusions from this case, basing them on this one experience and on sound child psychology.

First: Children who do poor school work, are apt to grow into an inferior state of mind, in which they accept tacit defeat from any subject to be attacked. This state of mind is caused, largely, by the driving, brow-beating type of teacher who persistently tries to make the erring pupil feel small and foolish.

Second: It is quite true that "Nothing succeeds like success." Since success is what we are seeking for our pupils, it is well for us to remember that one of the best ways to insure success is to expect it and to encourage the worker to attain it.

Third: It is wrong for a teacher to base his outward expectations of what quality of work a pupil is going to do, on his past records.

Fourth: I believe that there are numerous other cases similar to the one I have mentioned, and I for one would like to know about them through the columns of the News Letter.

MATHEMATICAL ABILITIES ARE COMPLEX

By DORA M. FORNO,
New Orleans, La.

Modern writers on Mathematics have stressed the fact,

which has been scientifically proven by psychologists and educators, that mathematical ability is not confined to the acquisition of one skill or technic, but includes a number of abilities more or less specific. Training in one ability does not produce results in all types, although it does in some respects influence some other type. For example, the solution of problems can never be effective unless all the underlying processes are understood and all basic facts are automatic.

I am not attempting to expound any new educational theory, but wish to emphasize some points of vital significance both in theory and practice. Regardless of all the expert knowledge offered us by psychologist and specialists in our field of learning, comparatively few seem to take to heart the suggestions made, and attack the teaching of mathematics scientifically. Convincing proof of these facts is evident in almost every class room, for wide variations in degrees of skill in every process and in the applications thereof are manifested generally.

All defects cannot be cured by remedial measures, for, in many cases, bad habits are so fixed that remedial measures will fail. My message is one of prevention.

My observation has shown without a doubt that, as the pupil progresses from grade to grade, it may be assumed that he knows the subject-matters of the preceding grade to a fair degree, but we halt with a jolt every now and then, when we find that much that we assume to be known, is wholly an unknown quantity. We assume too much, hence we should attack our problems more scientifically. Suppose we take an inventory of our pupils' knowledge of the preceding work since that is the basis of succeeding work in the grade. Find out what are class difficulties and by intensive systematic drill endeavor to overcome them. Then, endeavor to get the cooperation of the pupils to want to overcome their own difficulties in order to reach the standard of the group. The time taken to bring the class up to standard in some fundamental work will be justified by the intelligent approach to the new work.

The teaching procedure should be so organized that it purposes to have the children achieve an understanding of funda-

mental principles rather than the solution of individual problems. Having formulated such an aim, the utmost endeavor should be made to achieve that aim and test results by that measure. Nothing less than mastery of a fundamental process should be the starting point of a new process.

Every process in mathematics is based on some more fundamental process which must be fully grasped, if the more complex process is to be mastered. Now, we must not assume that any one process is simple, for every process is complex to the beginner and its mastery depends, not upon one ability only, but frequently upon many abilities.

If every mathematical ability were analyzed into its constituent abilities, it would be a great contribution to the pedagogy of mathematics. Textbook writers and teachers are endeavoring to do this to some degree, but much greater emphasis must be placed upon such analyses.

In my class in the psychology of arithmetic last summer at Tulane University, analyses of a number of the fundamental processes into the abilities involved were made and these revealed some interesting facts regarding the complexity of some of the processes.

These analyses of the processes into their constituent abilities are basic and should not be left to chance, but given thoughtful consideration before the presentation of a new process.

Here are a few of these analyses to illustrate the complexity of some of the so-called simple processes.

Abilities Involved in Long Division

1. Automatic control of the 90 primary facts of division.
2. Ability to give quick response to division with remainders.
3. Mastery of short division.

   (a) Where no carrying is involved.

   (b) Where carrying is involved.

   (c) Correct placing of quotient figures, especially where the division is not contained and a naught must be placed in product.

   (d) Names of the terms.

   (e) Proof.

4. Technique of long division .

A. New form of placing the quotient on top of the dividend and reason for so placing it.

B. Ability to see that the same processes are employed in long division as in short division.

1. Find little dividend.
2. Divide.
3. Multiply.
4. Subtract.
5. See that remainder is less than the divisor.
6. Bring down and continue process as before.

C. Ability to estimate the quotient figure correctly making a mental estimate before writing it in quotient.

D. Ability to make proper change in quotient figure if product is too large.

E. Ability to make proper change in quotient figure if remainder is too large.

F. Ability to place quotient figures in proper place in quotient, especially when the divisor is not contained in the partial dividend and a naught must be placed in the quotient.

G. Ability to express remainder as a fractional part of divisor and write it in quotient.

H. Ability to read quotient correctly expressing it in quantitative terms according to type of problem illustrated:

(1). Separating or partition division.

(2) Measuring or quotition division.

I. Ability to check division.

(1) When there is no remainder quotient $\times$ divisor $=$ dividend

(2) When there is a remainder, quotient $\times$ divisor $+$ remainder $=$ dividend.

Abilities Involved in Multiplication of Decimals

Multiplication of decimals depends upon the acquisition of approximately fourteen abilities.

I. Multiplication of whole numbers, which is a composite of the following:

A. Knowledge of the one hundred multiplication facts up to 9x9.

B. Ability to multiply two (or more)-place numbers by 2, 3, and 4, when carrying is not required and no zeros occur in the multiplicand.

C. Ability to multiply by 2, 3 ...... 9, with carrying.

D. Knowledge of the names of the terms used in multiplication; multiplicand, multiplier, and product.

E. Ability to multiply with two-place numbers not ending in zero.

F. Ability to handle zero in the multiplier as the first figure.

G. Ability to multiply with three (or more)-place numbers not including a zero.

H. Ability to multiply with three-and four—place numbers with zero in the second or third place, or second and third place, as well as in the first place.

I. Ability to multiply by 10, 100, and 1000 by annexing 1, 2, or 3 zeros respectively to the multiplicand to form the product; e. g.

$$250 \times 10 = 2500$$
$$25 \times 100 = 2500$$
$$25 \times 1000 = 25{,}000$$

II. Ability to multiply with United States money. This is taught before teaching multiplication of decimal fractions.

III. With the above abilities, the multiplication of decimals involves the ability to point off correctly in the product. Children must be taught that we point off in the product as many decimal places as there are in both the multiplicand and the multiplier. If the product does not contain as many figures as there are in the multiplicand and multiplier together, the necessary zeros must be prefixed to the product.

IV. The ability to multiply decimals by 10, 100 and 1000. The knowledge to move the decimal point one place to the right when multiplying by 10, two places when multiplying by 100; 100, and 1000, three places when multiplying by 1000. 1000 if the product does not three places when multiplying by 1000.

The knowledge to annex the necessary number of zeros when multiplying by 10, 100, and 1000 if the product does not contain enough figures.

Abilities Involved in Addition of Decimals

I. Abilities developed in earlier phases of arithmetic which form the background for addition of decimals.

A. Knowledge of the one hundred addition facts.

B. Ability to add whole numbers with precision in column with or without carrying.

C. Ability to write and add numbers of varying sizes from dictation.

II. New abilities developed by the teaching of "addition of decimals."

A. Names of different coins used.

B. Relative value of these coins to each other.

1. Ten dimes make a dollar, etc.
2. One dime is one-tenth of a dollar.
3. Ten is the basis of our money system.

C. Use of the decimal point to separate part of a dollar from the whole with whole to left of decimal point, and the part to the right of decimal point.

D. Writing of money values in straight columns with decimals under each other.

E. Addition of money values without carrying with decimal point in answer directly under decimals in addends.

F. Addition of money values with carrying from dimes to dollars.

G. Learning that whole numbers, not representing money values are based on ten.

H. Writing of parts of number that have the relative value of ten in same manner as dimes and cents; that is, to right of decimal point.

I. Meaning of the decimal names; tenth, one out of ten; hundredths, one out of a hundred, etc.

K. Writing of decimals in columns for adding without carrying.

L. Adding of decimals with carrying from tenths' to ones' column.

M. Addition of decimals with carrying in all columns.

N. Writing and adding decimals from dictation.

Abilities Involved in Division of Fractions

I Learning to divide a unit fraction by a whole number.

(a) Concrete assocation of unitary fractional division by use of the fraction chart. Measuring $\frac{1}{2}$ of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$.

(b) Find a shorter way. Learn that $\div 2$ is the same as $\times \frac{1}{2}$. Apply this to $\frac{1}{2} \div 2 =$. Think what was done to divide by the number;—the fraction was multiplied by the inverted divisor. Learn that the inverted number is the reciprocal of the number.

II Apply I to the division of any simple fraction by a whole number.

$4/5 \div 2 = \frac{1}{2} \times 4/5 = 2/5.$

III Learn to divide a fraction by a fraction.

(a) Fractions with like denominators $6/4 \div \frac{3}{4} =$

1. Compare $6/4 \div \frac{3}{4}$ to $4/3 \times 6/4$
2. Learn that the denominators will cancel.
3. From this find the short way—dividing the numerators.

(b) Fractions with unlike denominators $2/3 \div 5/7$.

Work it the known way—changing to L. C. D.; $14/21 \div 15/21$.
Apply III. $14/21 \div 15/21 = 14/15$
Find a shorter way by applying I.
Think: Dividing by $5/7$ is the same as multiplying by the reciprocal of $5/7$, e.g., $2/3 \div 5/7 = 7/5 \times 2/3 = 14/15$.

WRITTEN CLASS AND HOMEWORK

By RUTH I. PETTIGROVE
Lafayette School, New Orleans, La.

Written class work and homework should be of such a nature as to keep the individual pupil willingly occupied and to enlist the co-operation of all pupils.

We, as teachers, have found the majority of pupils below the standards of accuracy and speed because their habits of work are incorrect or because they lack skill in handling facts or mechanical processes. Speed without accuracy is useless.

We must be careful to give the child who is inaccurate or slow, adequate time to develop correct habits of work and mastery of the facts and fundamental processes.

Those habits which produce accuracy usually produce satisfactory speed. There is a speed at which each child does his best work. If he works below his natural speed, he wastes time and makes many errors. If he is hurried, he is tempted to work too rapidly, and again makes many errors. Just a little daily drill throughout the school session, to see how many *correct answers* the individual can get during the first five minutes of the Arithmetic period, will lay the foundation stone for future accuracy and speed with the fundamentals.

The work assigned should include diagnostic tests, preceded by development drills and followed by remedial drills and reviews. This material should be based on the principle of skill through repetition", and should conform to the laws of "learning and forgetting."

If we strive to establish correct habits of work, the pupil will willingly correct his own errors and we will undoubtedly avert much of the discouragement and dislike of Arithmetic. We can arouse more effort, interest and achievement through the motivation of team contests. Each member should help his team win by careful practice and accurate work, and develop the spirit of fair play and good sportsmanship, as he earnestly strives to make his team the winner.

The graph which the individual pupil keeps of his own record will encourage him to beat his own record, and make of himself a more efficient worker.

The five-minute-daily drill is one important division of classroom work. Correction of the homework, and assignment of the morrow's homework is another important part.

Home is the place to practice. Too often we assign homework of such a nature that it is a stumbling-block to the child. Drill exercises, papers to be corrected, and problems that can arouse an interest in the child are types of homework that are best. As to the proper amount of homework the teacher must remember not to overtax, but rather, to give a small assignment and to be sure that it is done to the best of the child's ability. All homework should be checked by the child himself, after it has been written and corrected on the board, or as the results are called, if it is merely a drill in fundamentals.

Another specific division of classroom work is the specific

grade-assignment which is given to pupils at the board and to workers at the desks. The classroom is the place to learn and to practice until the majority of pupils have grasped the lesson which has been developed. Each individual's work should have been carefully supervised.

Problems involving simple steps in solution are best in developing the pupil's ability to read and solve the fundamental types of problems. Many pupils have considerable difficulty in reading and solving even the simplest problems in multiplication and in division. If, in the "Facts Given" and in the "Question," pupils will write in words, "cost, weight, price, area, speed, volume, rate of gain, principal, etc.", he wil learn to solve a great many apparently difficult problems, under a few fundamental principles, as follows:

Number of articles $\times$ Price of 1 = Cost.
Time $\times$ Speed = Distance.
Length $\times$ Width = Area.
Base $\times$ Rate = Percentage.
Length $\times$ Width $\times$ height = Volume.
$T \times R \times P$ = Interest.

Pupils must memorize these principles and must understand the idea of factors and product, an understanding which is fundamental in the solution of all problems requiring multiplication and division.

Our aim in assignment of problems for classwork should be to give abundant practice in applying the few principles which are the key to all ordinary problems in arithmetic. Each pupil's success depends upon his ability to understand and use the facts and questions presented in the problem, upon his skill in working accurately, addition, subtraction, multiplication, or division of integers, decimals, or fractions, upon his faithfulness in checking results, and in deciding if his result is reasonable.

After days of practice, during which time the teacher should help and encourage each pupil, rather than mark down and discourage him, comes the actual testing time. The little subject or perhaps weekly tests ought to be given as an important part of the written work. These tests reveal to each pupil, his success or errors, and to the teacher, give an opportunity for

further necessary individual help along particular lines before the monthly test is given. The monthly test is a most necessary part of written work,—a summary,—a climax, which truly reveals the pupil's ability and attainment.

In conclusion may I tell of my humble efforts in my own classroom work?

When the class enters the room, homework papers are placed on desks for inspection, as the Drill Books ("Lennes Test and Practice Sheets in Arithmetic" published by Laidlaw Bros., Chicago) are distributed by regularly appointed leaders.

At a given signal, the class begins work on the page assigned, each pupil working at his own rate of speed on the Development or Remedial Drills. As each assignment is checked after a "stop signal" has been given, the pupil records his attainment on his graph, and books are returned to the cabinets.

To save time, during the five-minute drill period, a few pupils are selected to write their own efforts at homework on the board. A child is expected to do his homework to the best of his ability. If he does not succeed, he must bring in at least two trials. He knows that *home* is the place to practice, and that he does not receive any marks for this work. This effort is for his own good. Should he fail to bring his homework or a good excuse instead, he must return after school hours, and do more than was assigned. Few are guilty of this offense.

After the homework has been corrected or approved by the pupils, the new assignment is written on the board and quickly copied.

About 15 minutes time has been well spent, and the class is ready for the seat-work assignment which is written on the board. A group of from 8 to 12 pupils are assigned individual problems or statements to be solved on the board, as the remainder of the class begin work at their seats. The board assignments may be selected from the textbook, according to a basic principle previously taught, or may be from similar problems clipped from other arithemetics or papers, or original problems of pupils.

All pupils work busily for about 10 mnutes, as the teacher carefully supervises the work at the desks. The last 10 minute-period is given for class correction of board work. Then, should there be time before the departmental bell rings, papers are col-

lected and commented upon. Some are copied on the board, if correct, others are returned for correction.

These few suggestions I offer as my honest effort in the classroom—having once been a pupil who was led and guided by those who encouraged and inspired me to do my very best.

RELATIVE PRIMALITY

By S. T. SANDERS,
Louisiana State University

The prime is an integer having no integral divisor less than itself except unity. A primitive number (root) r is a number having the property that $r^n=1$ while r^k is not 1, k less than n, k and n positive integers. Such number, r, is called a primitive n^{th} root of unity. Let p be any prime number and let r be any one of the primitive n^{th} roots of 1, n being any postitive integer. If we restrict the factorization of p to the field of rational numbers, we have

$$(1) \qquad p=p.1$$

but, if we permit factorization in the field of complex numbers, we have

$$(2) \qquad p=p.r.r.\ldots\ldots\ldots r,$$ r being taken n times. If the integer n is chosen a prime, the fact that there are n-1 choices for the value of r, makes it evident that there must be at least (n-1) different ways in which p may factored, each way being associated with only two distinct factors, unless 1 should also be regarded as a factor.

Again, we might write

$$(3) \qquad p=p.r.r^2.\ldots\ldots\ldots r^k\ldots . r^{n-1}.$$

A well known theorem states that if r is any one of the primitive n^{th} roots of unity, then r^k is also a primitive n^{th} root, provided only that k is relatively prime to n. But, as n is an absolute prime, by hypothesis, it follows that every power, r^k, $k=1, 2, \ldots\ldots, n-1$, is a primitive n^{th} root of 1, since each k is relatively prime to n.

That (3) is true follows from the fact that when the r's are actually multiplied out the sum of the resulting exponents is the sum of the series

$1+2+\ldots\ldots+n\text{-}1=n(n\text{-}1)/2$, so that

$$\begin{aligned} p &= p(r^n)^s \\ &= p.(1)^s \\ &= p.1, \end{aligned}$$

where $s=n\text{-}1/2$.

Thus (3) shows a method of factoring the prime p into n distinct numbers taken from the complex field. As before, r is merely one of the n-1 primitive n^{th} roots of unity. But while, formally, any power, r^k, may be substituted for r in (3), proper reduction will show that the latter identity will be unaltered, though the order of the various powers of r in that case will be changed. If any two powers of r in (3) are multiplied or divided, whether they are distinct or identical, the resulting product or quotient is one of the powers, or, else it is unity. Manifestly, these two properties of the n^{th} roots of 1 *where n is any prime* may be used in the construction of a DEFINITION of the factorization of any prime p in the FIELD determined by the roots of $x^n—1=0$

In this sense the factorization (3) is unique. Assuming such a definition, an interesting parallelism becomes evident.

Corresponding to the fact that in the rational field every integer N is resolvable into a unique set of rational primes is the fact that in the field defined by the roots of $x^n—1=0$, where n is an arbitrary rational prime, each of these rational primes is itself in turn resolvable into a unique set of PRIMITIVE numbers. Since n is an aribitrary prime, if we identify n and p, (3) becomes (3)′ $p=p.r.r^2.\ldots\ldots.r^{p-1}$

In view of the above there would appear to be no logical objection to calling the exponents of r in (3)′ the primitives of p, or p-primitives, a mode of descripition which, manifestly, is more convenient than the language "integers relatively prime to p and less than p." Thus, the problem of determining the number of integers which are relatively prime to a given number n becomes identical with the problem of determining the number of the primitive n^{th} roots of unity.

But relatively primality has an interesting connection with absolute primality. If all integers less than N are relatively prime to a given number N, N is an absolute prime. If not,

N is only a relative prime. Expressed otherwise, if N is not an absolute prime, some of the N^{th} roots of unity are primitive and some are not. If N is an absolute prime, the number of its N-primitives is N-1, while if the contrary is true the number of N-primitives is less than N-1.

We shall presently establish a formula for the number of N-primitives in the latter case. It will be shown that in a great many cases the formula may be used to determine the number of *absolute primes* which are less than N, due to the fact that among the N-primitives are those absolute primes which are less than N but not contained in N.

The following theorems are used in our study.

Theorem 1. If r is any primitive n^{th} root of unity, then the n roots are $r, r^2, r^3, \ldots\ldots r^{n-1}, r^n=1$, and, those powers of the set whose exponents are relatively prime to n are the other primitive n^{th} roots of unity.

A proof of this will be found in Dickson's "First Course in the Theory of Equations."

Theorem 2. If $n=a.b$, where a and$b$ are any two factors of n, then any root of $x^a—1=0$ is also a root of $x^{ab}—1=0$.

Proof: If r is any root of $x^a—1=0$,

we have $\quad r^a—1=0,$

or, $\quad r^a \quad =1,$

whence, $\quad (r^a)^b=1,$

which is a condition that r shall be a root of

$$x^{ab}—1=0$$

Theorem 3. If there are p primitive n^{th} roots of unity, where $n=bc$, then the number of primitive N^{th} roots of unity, where $N=b^2c$, is bp.

Proof: All of the bc roots of $x^{bc}—1=0$ are, by Theorem 2, also roots of $x^{kc}—1=0$, where $k=b^2$.. Furthermore, by the definition of imprimitive roots of unity, they are also imprimitive roots of the latter equation.

Let R be any one of the p primitive roots of $x^{bc}—1=0$. By Theorem 1, there are p numbers of the sequence R, $R^2, \ldots\ldots,$ $R^{bc}=1$ whose exponents are relatively prime to $b.c$ and $(n-p)$ numbers of the same sequence whose exponents have a factor in common with $b.c$. Let R^e be an arbitrarily selected member of

the latter set, that is, R^e is *any* impritive root of $x^{bc}-1=0$. Then, identical with R^e is some imprimitive root r^s of $x^{kc}-1=0$, where $k=b^2$, and where r is a primitive root of the latter equation, so that s also has a common factor with b^2c *and* bc.

For each such imprimitive root R^e of $x^{bc}-1=0$, there is a set of imprimitive roots of $x^{kc}-1$, where $k=b^2$, namely,

$$r^s,\ r^{2s},\ \ldots\ldots,\ r^{bs}.$$

This is true for the following reasons: First, if s has a factor in common with b^2c, the exponents $2s, \ldots\ldots, b.s.$ have the same common factor. Second, if s is less than $b.c$, bs is less than b^2c and hence is one of the exponents of r in the series of powers of r which represent all the roots of $x^{kc}-1=0$. That s is less than bc is true because r^s, being equal to R^e, an imprimitive root of $x^{bc}-1=0$, must have a place among the first bc roots of $x^{kc}-1=0$, which, altogether, has b^2c *roots.*

Denoting the number of imprimitive roots of $x^{kc}-1=0$ by I, and the number of imprimitive roots of $x^{bc}-1=0$ by i the number of primitive roots of the former equation by P, those of the later by p, we have

$$N=b.n$$
$$I=b.i$$
$$n-i=p$$
$$N-I=p.$$

Also, $$N-I=b(n-i)$$
$$=b.p$$

Whence, $$P\quad=b.p,$$ *which* proves the theorem.

Theorem 4. There are $p-1$ primitive p^{th} roots of unity, if p is a prime number.

This is true from Theorem 1.

Theorem 5. If $n=a.b.\ldots\ldots.i$, in which, a, b, etc., are distinct prime factors, the number of primitive n^{th} roots of 1 is

$$(a-1)(b-1),\ldots\ldots(i-1).$$

Proof: By Theorem 4, $a-1$ is the number of primitive a^{th} roots of unity, $b-1$ is the number of primitive b^{th} roots of unity, etc.

The entire set of primitive roots thus described may be arranged in the following manner, $r, s, \ldots\ldots, v$, being themselves primitive roots. That is, r is a primitive a^{th} root, s a primitive b^{th} root of unity, etc.:

(1) $r, r^2, \ldots\ldots\ldots, r^{a-1}$,
(2) $s, s^2, \ldots\ldots\ldots, s^{b-1}$,
. . . .
. . . .
(k) $v, v^2, \ldots\ldots\ldots, v^{i-1}$

The product of any k of these numbers, so chosen that each of the k sets is represented in the product, is a primitive root of

$$x^{ab\ldots\ldots i}-1=0$$

For, suppose such product to be:

$$r^{j}\cdot s^{l}\cdot\ldots\ldots\ldots v^{m}$$

Since $r^{ja}=1, s^{lb}=1, \ldots\ldots, v^{mi}=1$, and since no exponents less than $a, b, \ldots\ldots\ldots, i$, wil satisfy these relations, it follows, as the latter numbers are prime, that

$$(r^{j}\cdot s^{l}\cdot\ldots\ldots\ldots v^{m})^{K}=1, \text{ where } K=a.b\ldots\ldots\ldots i,$$

Finally, no power of the bracketed product which is less than K can be equal to 1.

As there are $(a-1)(b-1)\ldots\ldots\ldots(i-1)$ such products it follows that there are $(a-1)(b-1)\ldots\ldots\ldots(i-1)$ primitive n^{th} roots of 1 where $n=a.b\ldots\ldots\ldots\ldots i$.

This proves the Theorem.

Theorem 6. If $n=p^a.q^b\ldots\ldots\ldots\ldots v^i$, in which p, q, etc., is are distinct primes, the number of primitive n^{th} roots of 1 is expressed by

$$P=(p^{a-1}.q^{b-1}\ldots\ldots\ldots\ldots v^{i-1})(p-1)(q-1)\ldots\ldots\ldots(v-1)$$

The truth of this Theorem follows immediately from Theorems 3 and 5.

Example 1. Find the number of integers relatively prime to 105 and less than 105 (i.e., the number of N-primitives, N=105)

By Theorem 5, denoting the number of N-primitives by P, we have, since N=3.5.7,

$$P=(3-1)(5-1)(7-1)$$

or

$$P=48.$$

Example 2. Find the number of N-primitives, if N is equal to $3^2.5^2$. We have,

$$N=3^2.5^2$$
$$=225$$

Hence, $P=(3.5)(3—1)(5—1)$, by Theorem 6,
$=15.2.4$
or, $P=120$

Thus, there are 120 integers less than 225 that have not the factors 3,5. (Actual count checks with this number, 1 being included among the primes.)

Example 3. If $n=2^3.5^2=200$, how many integers less than 200 are relatively prime to 200?

By Theorem 6,

$P=(2^2.5)(2—1)(5—1)$
or, $P=(20)(4)$
or, $P=80$

Actual count shows that there are 80 integers less than 200 which contain neither 2 nor 5 as a factor, 1 being included.

Example 4. If $N=2^4.7$, how many N-primitives are there?

We have, $P=2^3(2—1)(7—1)$, by Theorem 6.
or, $P=8.6$
or, $P=48$

Actual count shows 48 integers less than $2^4.7$ which are relatively prime to that number.

In some of the cases where n is not too large, Theorems 5 and 6 may be used to determine the number of absolute primes which are less than an assigned integer, n.

Example 5. Determine the number of absolute primes less than $n=2.3.5.7.=210$.

Since $n=2.3.5.7$, we have, by Theorem 6,
$P=1.2.4.6$
or, $P=48$

Manifestly, among these 48 integers which are relatively prime to 210 and less than 210 must be all absolute primes which are less than 210. It follows that if C is the number of composite relative primes in P, that $P—C+4$ is the number of absolute primes less than 210.

We now compute C.

The next prime after 7 is 11. Since no prime number greater than 19 can be multiplied by 11 to yield a product less than 210, all possible composite (relative) primes to 210 must contain only the primes 11, 13, 17, 19.

Inspection shows that the only composites satisfying this condition are:

11^2, 13^2, 11.13, 11.17, 11.19.

Thus, C=5, so that the number of absolute primes which are less than 210 is expressed by

$$A = 48-5+4=47.$$

Actual count verifiies the accuracy of this result.

Theorem 7. The number of $2n$-imprimitives is 2 times the number of n-imprimitives, where $n=2^k.a.b\ldots\ldots\ldots i$, $a, b, \ldots, i$ being primes.

Proof. By Theorem 6 the number of n-primitives is

$$2^{k-1}(2-1)(a-1)\ldots\ldots\ldots(i-1)$$

By the same Theorem the number of $2n$-primitives is

$$2^k(2-1)(a-1)\ldots\ldots\ldots(i-1).$$

This proves the Theorem.

Example 6. Determine the number of absolute primes which are less than $2^2.3.5.7=420$.

Example 5 showed that there are 48 integers relatively prime to and less than 2.3.5.7=210. Hence, by Theorem 7 there are 96 integers less and than relatively prime to 420, 48 of which integers must lie between 210 and 420, and furthermore are made up of, (*a*), absolute primes, (*b*), composite integers not having any one of the factors 2, 3, 5. 7.

Denote the number of these composite integers by C. Then 48—C will be the number of absolute primes between 210 and 420, and we proceed to compute C.

The smallest prime factor of any number in the C-set is 11, and the largest is, approximately, 420/11, which is 37.

Consider, then, the series of primes:

(1) 11, 13, 17, 19, 23, 29, 31, 37.

All those products or powers of the numbers of (1) which lie between 210 and 420 compose the C-class of integers. They are: 11.23, 11.29, 11.31, 11.37, 13.17, 13.19, 13.23, 13.29, 13.31, 17.17, 17.19, 17.23, 19.19.

Thus, C=13, 48—C= 48—13= 35, which is the number of absolute primes between 210 and 420.

Hence there are 47+35, or 82 absolute primes which are less than 420 if we count 1 as one of them.